

# INTERIOR REGULARITY ON THE ABREU EQUATION

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ABSTRACT. In this paper we prove the interior regularity for the solution to the Abreu equation in any dimension assuming the existence of the  $C^0$  estimate.

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## 1. INTRODUCTION

Studying extremal metrics within a given Kähler class on Kähler manifolds is one of the most important problems in complex geometry. Since Calabi introduced the notion of extremal metrics, it has been studied intensively in the past decades. There are three aspects of the problem: sufficient conditions, necessary conditions and the uniqueness of the existence. The necessary conditions for the existence are conjectured to be related to certain stabilities. There are many works on this aspect ([T], [D1],[D2],[CT]). For example, in [D1], Donaldson proved that the existence of Kähler metrics of constant scalar curvature implies the Chow or Hilbert (semi)stability. The uniqueness is completed by Donaldson, Mabuchi and Chen-Tian. In [D1], Donaldson proved that Kähler metrics of constant scalar curvature are unique in any rational Kähler class on any projective manifolds without non-trivial holomorphic vector fields. The complete answer is answered by Mabuchi for the algebraic case and answered by Chen-Tian([CT]) for

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general cases. They showed that Kähler metrics of constant scalar curvature are unique in any rational Kähler class on any compact Kähler manifold.

On the other hand, there has been few progresses on the existence of extremal metrics or Kähler metrics of constant scalar curvature. One reason is that the equation is highly nonlinear and of 4th order. The general expectation of the problem may be stated in the following conjecture ( see [D3]):

**Conjecture** A smooth polarised projective variety  $(V, L)$  admits a Kähler metric of constant scalar curvature in the class  $c_1(L)$  if and only if it is  $K$ -stable.

Donaldson has initiated a program on the extremal metrics on toric varieties. Since a  $2n$ -dimensional toric variety can be represented by a convex polytope in  $\mathbb{R}^n$ , the problem can be formulated on this polytope. In [A], using Gullimin's method([G]), Abreu reduced the equation for Kähler metrics of constant scalar curvature to an equation on the polytope. The equation is now called the Abreu equation. On the other hand, Donaldson([D3]) formulated the problem as an variational problem on the polytope and related it to the  $K$ -stability. Hence, the problem is to solve the Abreu equation with the assumption of the stability condition. Since the equation is a degenerate 4th order PDE, it is a very hard problem and the progress is slow. We list some important progresses obtained by Donaldson: (1) in [D3] Donaldson introduces a stronger version of stability (cf. Proposition 5.1.2 and Proposition 5.2.2 in [D3]); Donaldson proved the existence of weak solutions when this stronger stability holds; (2) in [D4], he proved the interior estimates for the Abreu equations on toric surfaces; (3) in [D5], he proved the conjecture for the toric surfaces under the assumption of  $M$ -condition, and (4) in [D6], he completely solved the conjecture for the constant scalar curvature metrics on toric surfaces. On the other hand, in [ZZ], Zhou-Zhu proved the existence of weak solutions under certain properness condition.

In this paper, we prove the interior estimates for the solution to the Abreu equation in any dimension assuming the existence of the  $C^0$  estimate. Be precisely, the statement is

**Theorem 1.1.** *Let  $M$  be a real  $2n$ -dimensional compact toric manifold and  $\Delta$  be its Delzant polytope. Let  $K \in C^\infty(\overline{\Delta})$ . If  $u$  is a solution to*

the Abreu equation  $\mathcal{S}(u) = K$  (cf. (2.2)), i.e., it gives a metric on toric variety of curvature  $K$ , then for any domain  $\Omega \subset\subset \Delta$  and positive integer  $k$ , there exists a constant  $C$  depending on  $n, |K|_{C^m(\overline{\Delta})}, |u|_{C^0(\overline{\Delta})}$  and  $d := \text{dist}(\Omega, \partial\Delta)$  such that

$$(1.1) \quad |u|_{C^{k,\alpha}(\Omega)} \leq C(n, |K|_{C^m(\overline{\Delta})}, |u|_{C^0(\overline{\Delta})}, d),$$

where  $m = \max\{n - 3, 0\}$ .

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## 2. TORIC MANIFOLDS AND THE ABREU EQUATION

A toric manifold is an  $2n$ -dimensional symplectic manifold  $(M, \omega)$  admitting a  $T^n$  Hamiltonian action. The image of the moment map of the Hamiltonian action is then a polytope in  $\mathbb{R}^n$ . Such polytopes are called Delzant polytopes. Be precisely, we have the following definition.

**Definition 2.1.** A convex polytope  $\Delta$  in  $\mathbb{R}^n$  is a Delzant polytope if

- (1) there are  $n$  edges meeting at each vertex ;
- (2) the edges meeting at the vertex  $p$  are rational, i.e. each edge is of the form  $p + tv_i$ ,  $0 \leq t \leq \infty$ , where  $v_i \in \mathbb{Z}^n$ ;
- (3)  $v_1, \dots, v_n$  described above can be chosen to form a basis of  $\mathbb{Z}^n$ .

A Delzant polytope can be defined by  $d$  linear inequalities

$$\langle \xi, h_k \rangle \geq \lambda_k, \quad k = 1, \dots, d,$$

where  $h'_k$ s are primitive elements of the lattice  $\mathbb{Z}^n$ . Set  $\ell_i : \mathbb{R}^n \rightarrow \mathbb{R}$  be the function  $\ell_i(\xi) = \langle \xi, h_i \rangle - \lambda_i$ , and let  $\Delta^\circ$  be the interior of  $\Delta$ . In this paper, we assume that  $\Delta = \Delta^\circ$ . Then  $\xi \in \Delta$  if and only if  $\ell_i(\xi) > 0$  for all  $i$ .

In [De] Delzant associates to every Delzant polytope  $\Delta$  a closed connected symplectic toric manifold  $(M_\Delta, \omega)$  of dimension  $2n$  together with a Hamiltonian  $T^n$ -action whose moment map  $\varphi : M_\Delta \rightarrow \mathbb{R}^n$  satisfies  $\varphi(M_\Delta) = \overline{\Delta}$ . On  $(M_\Delta, \omega)$ , Guillemin shows that there is a natural Kähler metric called the Guillemin metric, denoted by  $G_g$  or  $\omega_g$  (cf. (2.1)).

Let  $T = T^n$  and  $C_T^\infty(M_\Delta)$  be the set of  $T$ -invariant functions. Denote

$$\mathcal{M} = \{\phi \in C_T^\infty(M_\Delta) | \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \phi > 0\}.$$

For any  $\phi \in \mathcal{M}$ , restricting on  $\varphi^{-1}(\Delta)$ , we have a Kähler potential

$$f(x) = g(x) + \phi(x),$$

where  $g(x)$  is the Guillemin potential function for  $\omega_g$ . There are natural isomorphisms

$$\varphi^{-1}(\Delta) \cong (\mathbb{C}^*)^n \cong (\mathbb{R} \times S^1)^n \cong \mathbb{R}^n \times T,$$

then  $f$  and  $g$  are treated as functions on  $\mathbb{R}^n$  whose coordinates are still denoted by  $x_1, \dots, x_n$ . By the Legendre transformation, we put

$$\xi_i = \frac{\partial f}{\partial x_i}, \quad u(\xi) = \sum \xi_i \frac{\partial f}{\partial x_i} - f(x).$$

It is known that  $u(\xi)$  is a smooth function on  $\Delta$ . Similarly, the Legendre transformation of  $g$  also defines a smooth function  $v$  on  $\Delta$ . By Guillemin's theorem,  $v$  is given by

$$(2.1) \quad v(\xi) = \sum_{k=1}^d \ell_k(\xi) \log \ell_k(\xi).$$

For each  $u$ , set  $\psi_u = u - v$ . It is known that  $\psi_u$  is a smooth function on  $\overline{\Delta}$ .

We will use  $\mathcal{S}(f)$  or  $\mathcal{S}(u)$ , or even  $\mathcal{S}(\phi)$  to denote the scalar curvature for the Kähler metric  $\omega_\phi$ . The Abreu equation(see [D1]) is

$$(2.2) \quad \mathcal{S}(u) = \sum_{i,j=1}^n U^{ij} w_{ij} = -K \quad \text{in } \Delta$$

where  $K$  is a given function on  $\overline{\Delta}$ ,  $U^{ij}$  is the cofactor of the Hessian matrix  $D^2u$  of the convex function  $u$  and

$$w = [\det D^2u]^{-1}.$$

Set

$$\mathfrak{R}(\Delta) = \{u = v + \psi | \psi \in C^\infty(\overline{\Delta})\}$$

and

$$\mathfrak{R}(\Delta, b) = \{u \in \mathfrak{R}(\Delta) | |\mathcal{S}(u)| \leq b\}.$$

Let  $u$  be a function on  $\Delta$ . We say it is normal at  $p \in \Delta$  if  $u(p) = 0$  and  $\nabla(p) = 0$ . It is known that for any function and any point  $p \in \Delta$ , the function can be normalized by adding an affine function. Set  $\mathfrak{R}_p(\Delta) \subset \mathfrak{R}(\Delta)$ ,  $(\mathfrak{R}_p(\Delta, b) \subset \mathfrak{R}(\Delta, b))$  be the set of normal functions at  $p$ .

**Remark 2.2.** *In this paper,  $K$  can be any smooth function. We remark that the Kähler metric is extremal if and only if  $K$  is an affine function in  $\xi$ .*

### 3. DETERMINANT ESTIMATES

The key estimate in this paper is Proposition 3.1, which holds for any Kähler manifold other than toric manifolds.

Let  $(M, G)$  be a compact complex manifold with a Kähler metric  $G$ . Let  $\omega_G$  be its Kähler form. Denote

$$\widetilde{\mathcal{M}} = \{\phi \in C^\infty(M) \mid \omega_\phi = \omega_G + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0\}.$$

Choose a local coordinate system  $z_1, \dots, z_n$ , let  $g$  be a local potential function of  $G$  in this coordinate system. For any  $\phi \in \widetilde{\mathcal{M}}$ , let  $f = g + \phi$ . Denote

$$H := \frac{\det(g_{i\bar{j}})}{\det(f_{i\bar{j}})},$$

it is known that  $H$  is a global function defined on  $M$ . Set  $\mathcal{K} = \max_M \|Ric(g)\|_g$ , where  $Ric(g)$  denotes Ricci tensor for metric  $G$ . We prove

**Proposition 3.1.** *For any  $\phi \in \widetilde{\mathcal{M}}$  we have*

$$(3.1) \quad H \leq \left(2 + \frac{\max_M |\mathcal{S}(\phi)|}{n(\mathcal{K} + 1)}\right)^n \exp \left\{ (2\mathcal{K} + 1) (\max_M \{\phi\} - \min_M \{\phi\}) \right\}.$$

**Proof.** Consider the function

$$\mathcal{F} := \exp\{-C\phi\}H,$$

where  $C$  is a constant to be determined later.  $\mathcal{F}$  attains its maximum at a point  $p^* \in M$ . We have, at  $p^*$ ,

$$-Cf^{i\bar{j}}\phi_{i\bar{j}} + f^{i\bar{j}}(\log H)_{i\bar{j}} \leq 0$$

which implies

$$(3.2) \quad -Cf^{i\bar{j}}\phi_{i\bar{j}} + \mathcal{S}(\phi) + \sum f^{i\bar{j}}(\log \det(g_{k\bar{l}}))_{i\bar{j}} \leq 0.$$

We can choose the coordinates  $z_1, \dots, z_n$  at a neighborhood of  $p^*$  such that

$$f_{i\bar{j}} = \lambda_i \delta_{ij}, \quad g_{i\bar{j}} = \mu_i \delta_{ij}.$$

From (3.2), we get

$$C \left( \frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) - Cn + \mathcal{S} - \left( \frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) \mathcal{K} \leq 0.$$

We choose  $C = 2\mathcal{K} + 1$  and apply an elementary inequality

$$\frac{1}{n} \left( \frac{\mu_1}{\lambda_1} + \cdots + \frac{\mu_n}{\lambda_n} \right) \geq \left( \frac{\mu_1 \cdots \mu_n}{\lambda_1 \cdots \lambda_n} \right)^{1/n}$$

to get

$$n(\mathcal{K} + 1)H^{\frac{1}{n}} \leq n(2\mathcal{K} + 1) + |\mathcal{S}|.$$

It follows that, at  $p^*$ ,

$$\exp\{-C\phi\}H \leq \left( 2 + \frac{|\mathcal{S}|}{n(\mathcal{K} + 1)} \right)^n \exp\{-(2\mathcal{K} + 1) \min_M \{\phi\}\}.$$

Then (3.1) follows.  $\square$

On the other hand, on  $\Delta$ , Donaldson gives several estimates for  $u \in \mathfrak{R}(\Delta, b)$ .

**Lemma 3.2.** *Suppose that  $u \in \mathfrak{R}(\Delta, b)$ . Then  $\det(D^2u) \geq d_1$  everywhere in  $\Delta$ , where*

$$d_1 = \left( \frac{4b \text{Diam}(\Delta)^2}{n} \right)^{-n}.$$

This lemma can be found in [D4].

**Lemma 3.3.** *For any  $u \in \mathfrak{R}(\Delta, b)$ , we have*

$$\|\phi\|_{C^0(M)} = \|u - v\|_{C^0(\overline{\Delta})},$$

where  $\phi = f - g$ , and  $f$  is the Legendre transform of  $u$ .

This fact was already used by Donaldson in [D6], and for the proof, the readers are referred to the proof of Proposition 4.4 in [SZ].

#### 4. PROOF OF MAIN THEOREM

**Proof of Theorem 1.1.** Without loss of generality, we assume that  $v$  is normalized at  $0 \in \Delta$ , i.e.,

$$v(0) = 0, \quad \nabla v(0) = 0.$$

Then  $g$  attains its minimum at 0. Suppose that

$$(4.1) \quad \|u\|_{C^0(\overline{\Delta})} \leq \mathcal{C}_1, \quad \|v\|_{C^0(\overline{\Delta})} \leq \mathcal{C}_1$$

for some constant  $\mathcal{C}_1 > 0$ . Let  $f$  be the Legendre transforms of  $u$ . By Lemma 3.3 and (4.1), for any  $C > 4\mathcal{C}_1$ , we have

$$(4.2) \quad S_g(C - 2\mathcal{C}_1) \subset S_f(C) \subset S_g(C + 2\mathcal{C}_1),$$

where

$$S_f(C) = \{x | f(x) \leq C\}, \quad S_g(C) = \{x | g(x) \leq C\}.$$

By Lemma 3.2,

$$(4.3) \quad \det(f_{ij}) \leq d_1^{-1}.$$

On the other hand, by Proposition 3.1 we have

$$\frac{\det(g_{ij})}{\det(f_{ij})} \leq \mathcal{C}_2(n, b, \mathcal{K}, \mathcal{C}_1).$$

Restricting to  $S_f(C) \subset S_g(C + 2\mathcal{C}_1)$ , we have

$$(4.4) \quad \det(f_{ij}) \geq \mathcal{C}_2^{-1} \det(g_{ij}) \geq d_2(\mathcal{C}_2, C).$$

By the determinant estimates (4.3) and (4.4), applying the Caffarelli-Gutiérrez theory (see [CG]) and the Caffarelli-Schauder estimate (see [CC]) in  $S_f(C)$ , we conclude that

$$(4.5) \quad \|f\|_{C^{3,\alpha}(S_g(\frac{C}{2} - 2\mathcal{C}_1))} \leq \|f\|_{C^{3,\alpha}(S_f(\frac{C}{2}))} \leq \mathcal{C}_3(n, b, \mathcal{K}, \mathcal{C}_1, C)$$

for any  $C > 4\mathcal{C}_1$  (details see [TW]).

For any domain  $\Omega \subset\subset \Delta$ , by the convexity of  $u$  and (4.1) we have

$$\nabla u(\Omega) \subset B_{\frac{2\mathcal{C}_1}{d}}(0),$$

while the later one is contained in  $S_g(C_3)$  for some constant  $C_3 > 0$ . Here  $d = \text{dist}(\Omega, \partial\Delta)$  denotes the Euclidean distance from  $\Omega$  to  $\partial\Delta$ . Choose  $C = 2C_3 + 4\mathcal{C}_1$ . By (4.5), we obtain  $\|u\|_{C^{3,\alpha}(\Omega)} \leq \mathcal{C}_3(n, b, \mathcal{K}, \mathcal{C}_1, C)$ . The Proposition follows from the standard bootstrap argument.  $\square$

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